

**18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)**  
**Error bounds in the prime number theorem**

In this unit, we introduce (without proof for now) a formula which relates the distribution of primes to the zeroes of the Riemann zeta function. Given a suitable zero-free region for  $\zeta(s)$  in the critical strip, this can be used to prove the prime number theorem with an estimate for the error term.

## 1 Zeta zeroes and prime numbers

For  $x \notin \mathbb{N}$ , define the counting function

$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

where  $\Lambda : \mathbb{N} \rightarrow \mathbb{R}$  is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & n = p^a, a \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If  $x \in \mathbb{N}$ , it is convenient to modify the definition to

$$\psi(x) = \sum_{n < x} \Lambda(n) + \frac{1}{2} \Lambda(x).$$

Note that for the function  $\vartheta$  we defined earlier as

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

we have

$$\psi(x) - \vartheta(x) = O(x^{1/2} \log x) \quad (x \rightarrow \infty)$$

so the prime number theorem is equivalent to

$$\psi(x) \sim x \quad (x \rightarrow \infty).$$

The formula of von Mangoldt expresses the difference  $\psi(x) - x$  in terms of the zeroes of  $\zeta(s)$ . We will prove this formula in a later unit.

**Theorem 1** (von Mangoldt's formula). *For  $x \geq 2$  and  $T > 0$ ,*

$$\psi(x) - x = - \sum_{\rho: |\operatorname{Im}(\rho)| < T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}) + R(x, T)$$

*with  $\rho$  running over the zeroes of  $\zeta(s)$  in the region  $\operatorname{Re}(s) \in [0, 1]$ , and*

$$R(x, T) = O \left( \frac{x \log^2(xT)}{T} + (\log x) \min \left\{ 1, \frac{x}{T \langle x \rangle} \right\} \right).$$

*Here  $\langle x \rangle$  denotes the distance from  $x$  to the nearest prime power other than possibly  $x$  itself.*

The region  $\operatorname{Re}(s) \in [0, 1]$  is called the *critical strip* for  $\zeta$ , because we can account for all of the zeroes outside this strip: they are the trivial zeroes  $s = -2, -4, \dots$  forced by the functional equation and the fact that  $\Gamma(s/2)$  has poles at nonpositive even integers. In fact, the last term in the formula is merely  $-\sum_{\rho} \frac{x^{\rho}}{\rho}$  for  $\rho$  running over the trivial zeroes.

Incidentally, one can check by a numerical calculation that there are no real zeroes of  $\zeta$  in the critical strip, by numerically approximating the integral representation of  $\xi(s)$ . This raises an interesting point: in general, direct numerical approximation can be used to prove that an analytic function does not vanish in a region, but not that it does vanish at a particular point. The best one can do is use a zero-counting formula to prove that there must be a zero near the proposed vanishing point.

Note that for  $x$  fixed,  $R(x, T) = o(1)$  as  $T \rightarrow \infty$ , so we have

$$\psi(x) - x = -\sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2})$$

as long as we interpret the sum over  $\rho$  to mean the limit of the partial sums over  $|\operatorname{Im}(\rho)| < T$  as  $T \rightarrow \infty$ . This formula, while pretty, is not as useful in practice as the form with remainder; we will use the remainder form by taking  $T$  to be some (preferably large) function of  $x$  as  $x \rightarrow \infty$ .

## 2 How to use von Mangoldt's formula

In order to use von Mangoldt's formula to bound  $\psi(x) - x$ , we need to give an upper bound on the sum  $\sum_{\rho} x^{\rho}/\rho$  for  $\rho$  running over nontrivial zeroes of  $\zeta$  in the region  $|\operatorname{Im}(s)| \leq T$ .

Put  $\beta = \operatorname{Re}(\rho)$ ,  $\gamma = \operatorname{Im}(\rho)$ . Suppose we can prove that  $\beta < 1 - f(|\gamma|)$  for some nonincreasing function  $f : [0, \infty) \rightarrow (0, 1/2)$ ; then

$$|x^{\rho}| = x^{\beta} < x^{1-f(|\gamma|)} < x^{1-f(T)}$$

and  $|\rho| \geq |\gamma|$ . We thus have

$$\left| \sum_{\rho: |\gamma| < T} \frac{x^{\rho}}{\rho} \right| \leq x^{1-f(T)} \sum_{\rho: |\gamma| < T} \frac{1}{\gamma}.$$

Let  $N(T)$  be the number of zeroes in the critical strip with  $|\gamma| \leq T$ . Then

$$\sum_{\rho: 0 < |\gamma| < T} \frac{1}{\gamma} = \int_0^T t^{-1} dN(t) = \frac{N(T)}{T} + \int_0^T t^{-2} N(t) dt.$$

At this point we need some information about  $N(T)$ ; again, we will prove this (and a bit more) later.

**Theorem 2** (Hadamard). *We have  $N(T) = O(T \log T)$  as  $T \rightarrow \infty$ .*

This implies that

$$\left| \sum_{\rho: |\gamma| < T} \frac{1}{\gamma} \right| = O(\log^2 T),$$

so

$$\left| \sum_{\rho: |\gamma| < T} \frac{x^\rho}{\rho} \right| = O(x^{1-f(T)} \log^2 T).$$

For  $x$  an integer, we now take  $T = T(x)$  to be a suitable function of  $x$ , and invoke von Mangoldt's formula with remainder to deduce that

$$\psi(x) - x = O\left(x^{1-f(T)} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)}\right). \quad (1)$$

### 3 The Riemann Hypothesis

Riemann calculated a few of the zeroes of  $\zeta$  and, based on this evidence, made the following remarkable conjecture (whose resolution is worth \$1,000,000 from the Clay Mathematics Institute).

**Conjecture 3** (Riemann Hypothesis). *The nontrivial zeroes of  $\zeta$  all lie on the line  $\operatorname{Re}(s) = \frac{1}{2}$ .*

This is a best-case scenario in terms of deducing error bounds on  $\psi(x) - x$ . Namely, suppose every nontrivial zero  $\rho$  of  $\zeta$  satisfies  $c \leq \operatorname{Re}(\rho) \leq 1 - c$  for some  $c \in (0, 1/2)$ ; then we can take  $f(T) = c$  in (1), yielding

$$\psi(x) - x = O\left(x^{1-c} \log^2 T(x) + \frac{x \log^2 x}{T(x)} + \frac{x \log^2 T(x)}{T(x)}\right).$$

By taking  $T(x) = x$ , we obtain

$$\psi(x) - x = O(x^{1-c} \log^2 x).$$

If I can take  $c$  to be any value less than  $1/2$ , that means

$$\psi(x) - x = O(x^{1/2+\epsilon}) \quad (\epsilon > 0),$$

and similarly one gets a strong estimate on  $\pi(x)$  (see exercises).

Unfortunately, for *no* value of  $c > 0$  are we able at present to prove that every nontrivial zero  $\rho$  satisfies  $\operatorname{Re}(\rho) \leq 1 - c$ . We will give a much smaller zero-free region in a later unit.

## 4 Variants for $L$ -functions

For  $\chi$  a Dirichlet character, define

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n) \Lambda(n),$$

where again we multiply the  $n = x$  term by  $1/2$  if it is present.

**Theorem 4.** For  $\chi$  a nonprincipal Dirichlet character of level  $N$ ,

$$\psi(x, \chi) = - \sum_{\rho: |\gamma| < T} \frac{x^\rho}{\rho} - (1-a) \log x - b(\chi) + \sum_{m=1}^{\infty} \frac{x^{a-2m}}{2m-a} + R(x, T),$$

where  $b(\chi)$  is an explicit constant,  $a = 1$  for  $\chi$  even and  $a = 0$  for  $\chi$  odd, and

$$R(x, T) = O\left(\frac{x \log^2(NxT)}{T} + (\log x) \min\left\{1, \frac{x}{T\langle x \rangle}\right\}\right).$$

For a fixed  $N$ , one can use this formula together with a zero-free region for all of the  $L(s, \chi)$  with  $\chi$  of level  $N$ , to obtain a prime number theorem for arithmetic progressions of difference  $N$  with an estimate for the error term.

However, one would also like to be able to establish a prime number theorem with error term for arithmetic progressions where the difference is allowed to vary. In this case, one of course must have a zero-free region for all of the relevant characters. But there are two extra complications.

- One must understand how the constant  $b(\chi)$  varies with  $\chi$ .
- One must deal with possible roots of  $L(s, \chi)$  that are very close to  $s = 0$  or  $s = 1$  (so-called *Siegel zeroes*).

Dealing with these goes beyond the level of detail I have in mind for this course; see Davenport §14–22 for a systematic exposition.

## Exercises

1. Assume that  $\psi(x) = x + o(x^{1-\epsilon})$  for some given  $\epsilon \in (0, 1/2)$ . Deduce a corresponding upper bound for  $\pi(x) - \text{li}(x)$ , where  $\text{li}(x)$  is the logarithmic integral function

$$\text{li}(x) = \int_2^x \frac{dt}{\log t}.$$

Then deduce that

$$\pi(x) - \frac{x}{\log x} \neq o(x^{1-\delta})$$

for any  $\delta > 0$ . (This last statement can be proved unconditionally, but don't worry about that for now.) This is the sense in which  $\text{li}(x)$  is a better approximation than  $x/(\log x)$  of the count of primes.